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On a generalised G-function in radiative transfer theory of turbid vegetation media

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Summary

The simplified approach of a turbid medium is commonly applied in theory of radiative transfer for vegetation media. Oriented planar model leaves are assumed whose normals are always confined to the upper half space. These orientations are described with the help of so-called leaf normal distribution functions (LNDFs) so that, within the scope of the turbid theory, a radiative transfer equation can be derived in which the so-called Ross-Nilson function G occurs explicitly. This function, as introduced by J. Ross, is based on geometrical considerations and is therefore called geometry function, or shortly G-function (GF). To solve the latter equation G must be known. GF is calculated from the LNDF and was originally derived in an explicit and analytical form for strongly simplified LNDFs only. We demonstrated in a previous work that GF can be calculated also for other standard LNDFs. Based on the latter LNDFs we introduce here a generalised trigonometric LNDF and present the respective formula for G.

Zusammenfassung

Die vereinfachte Annahme eines turbiden Mediums findet in der Theorie des Strahlungstransfers für Vegetationsmedien breite Anwendung. Darin werden orientierte ebene Modellblätter angenommen, deren Normalen stets in den oberen Halbraum weisen. Diese Orientierungen werden mittels sogenannter Blattnormalenverteilungen (BNV) beschrieben, so dass sich im Rahmen der turbiden Theorie eine Strahlungstransfergleichung ableiten lässt, in der die sogenannte Ross-Nilson-Funktion G explizit auftritt. Diese von J. Ross eingeführte Funktion basiert auf geometrischen Betrachtungen und wird daher auch Geometriefunktion genannt oder kurz G-Funktion. G muss zur Lösung der vorigen Gleichung bekannt sein. Es leitet sich aus der BNV ab und konnte in expliziter sowie analytischer Form bislang lediglich für stark vereinfachte BNV hergeleitet werden. Wie wir an dieser Stelle in einem früheren Beitrag gezeigt haben, lässt sich G darüber hinaus für andere standardisierte BNV berechnen. Auf letzteren aufbauend führen wir jetzt eine verallgemeinerte trigonometrische BNV ein und präsentieren die entsprechende Formel für G.

1 Introduction

The assumption of a turbid medium in vegetation radiative transfer means to treat the model leaves to be planar having sizes much smaller than the dimension of the considered region of a vegetation canopy for which each point is represented by a collection of oriented model leaves. This is a strong simplification but, nevertheless, it is an often applied approach in radiative transfer theories for vegetation media. The directions of the leaf normals are assumed to be confined to the upper half space and are described by a leaf normal distribution function (LNDF) which is generally also a function of space and time.
Leaves intercept radiation that enters the canopy from the atmosphere above. To account for this extinction process a projection function was introduced in the 1960s (Nilson, 1971; Ross, 1981) which is primarily dependent on the direction of the radiation. This projection function is defined as the integral of the LNDF, projected onto the radiation direction considered, over the upper unit half sphere. Due to the geometrical aspect of a simple projection, this Ross-Nilson function G is also called 'geometry function', that is shortly, G-function (GF). The extinction coefficient is then written as a function of G, so that GF is an integral part of the radiative transfer equation that can be derived for a turbid vegetation medium (Ross, 1981; Marshak and Davis, 2005). To solve this equation analytically for a given LNDF, an explicit expression for the respective GF is required. However, during the last 40 years GF could only be determined in an explicit and analytical form for few very simplified cases, e.g., for purely horizontal model leaves. To account for more realistic canopies described by a more complex LNDF, the respective GF was either approximated, associated with a certain inaccuracy, or was calculated numerically accepting a certain computational effort. In the following we extend the theory by defining a generalised LNDF for more realistic canopy architectures and calculate the according GF explicitly and analytically.

2 LNDF and GF - standard cases

The *leaf normal distribution function* (LNDF) is a dimensionless function $g_L(\mathbf{x}, \mathbf{y}_L, t) \geq 0$ of space $\mathbf{x}$ as well as time t and is a distribution of the probability that the normal vector $\mathbf{y}_L$ of a local planar leaf is confined to the upper half space S_1^+ fulfilling the normalisation condition

$$1 = \frac{1}{A(S_1^+)} \int\limits_{S_1^+} g_L(\mathbf{x}, \mathbf{y}_L, t)\, do(\mathbf{y}_L) \quad \forall\, \mathbf{x}, t$$

where $A(S_1^+)$ is surface area of S_1^+. The latter can be parameterised by

$$\begin{aligned} S_1^+ &= \{\, \boldsymbol{\omega}(\vartheta,\varphi) \in \mathbb{R}^3 \,|\, (\vartheta,\varphi) \in [0,\tfrac{\pi}{2}] \times [0,2\pi] \,\} \\ &= \{\, \hat{\boldsymbol{\omega}}(\mu,\varphi) \in \mathbb{R}^3 \,|\, (\mu,\varphi) \in [0,1] \times [0,2\pi] \,\} \end{aligned} \tag{1}$$

with the spherical coordinates $\vartheta = \cos^{-1}(\mu)$ and φ (Otto and Trautmann, 2008a,b).

We assume that LNDF is homogeneous as well as independent of time, and we separate its angular dependence according to

$$\begin{aligned} g_{_L}(\mathbf{y}_{_L} = \hat{\boldsymbol{\omega}}(\mu_{_L},\varphi_{_L})) &= g_\mu(\mu_{_L})\, g_\varphi(\varphi_{_L}) \\ g_{_L}(\mathbf{y}_{_L} = \boldsymbol{\omega}(\vartheta_{_L},\varphi_{_L})) &= g_\vartheta(\vartheta_{_L})\, g_\varphi(\varphi_{_L}) \end{aligned} \tag{2}$$

in terms of the azimuth angle $\varphi_{_L}$ and the cosine $\mu_{_L} = \cos\vartheta_{_L}$ of the zenith angle of the leaf normal where $g_\mu(\mu_{_L}) = g_\vartheta(\cos^{-1}(\mu_{_L}))$. If we further let LNDF be distributed uniformly with respect to $\varphi_{_L}$, that is, $g_\varphi(\varphi_{_L}) := 1$, the following normalisation condition results

$$\int\limits_{[0,\frac{\pi}{2}]} g_\vartheta(\vartheta_{_L})\, \sin\vartheta_{_L}\, d\vartheta_{_L} = \int\limits_{[0,1]} g_\mu(\mu_{_L})\, d\mu_{_L} = 1. \tag{3}$$

This is the oftenly adopted approach in which only g_ϑ or g_μ represent a LNDF as a simple function of $\mu_{_L} \in [0,1]$. Table 1 presents analytical expressions of g_ϑ for standard LNDFs which describe predominantly horizontally oriented leaves.

Given a LNDF the *geometry function* (GF) is a dimensionless and positive function which is defined by

$$G(\mathbf{y}) := \frac{1}{A(S_1^+)} \int\limits_{S_1^+} |\langle \mathbf{e}_\mathbf{y}, \mathbf{e}_{\mathbf{y}_L} \rangle|\, g_{_L}(\mathbf{y}_{_L})\, do(\mathbf{y}_{_L}), \quad \mathbf{y} \in \mathbb{R}^3 \setminus \{\mathbf{0}\} \tag{4}$$

where $g_{_L} \neq f(\mathbf{x},t)$ was assumed as above and $\mathbf{e_z} := \frac{\mathbf{z}}{\|\mathbf{z}\|}$ is an unit vector in direction of $\mathbf{z} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$. The Euclidian scalar product $\langle \cdot\,,\cdot \rangle$ under the integral expresses the projection of $g_{_L}$ for a certain normal direction $\mathbf{y}_{_L}$ onto the particular direction vector $\mathbf{y}$ of the radiation. GF become larger the more the most probable directions of the leaf normals are confined into the radiation direction considered, or, in other words, the more the most probable leaf elements are perpendicular to $\mathbf{y}$. Thus, GF is the total probability for leaf orientations perpendicular to the radiation direction. Since interactions between the radiation and the leaf elements can be assumed to be the more probable the more the leaves are oriented perpendicularly to the direction of radiation, GF is a measure for extinction.

Table 1: Usually applied standard LNDFs g_ϑ arranged in descending order of increasing fractions of horizontal leaves (vertical leaf normals) together with the respective GFs after equation (5) as function of μ. For the according expressions of g_μ see Otto and Trautmann (2008a).

Notation	$g_\vartheta(\vartheta_{_L})$	$G_\mu(\mu)$
spherical-0	1	$\frac{1}{2}$
spherical-1	$2\cos\vartheta_{_L}$	$\frac{4}{3\pi}\left[\frac{\pi}{2}\lvert\mu\rvert - \lvert\mu\rvert\cos^{-1}(\lvert\mu\rvert) + \sqrt{1-\mu^2}\right]$
spherical-2	$3\cos^2\vartheta_{_L}$	$\frac{3}{8}(1+\mu^2)$
spherical-3	$4\cos^3\vartheta_{_L}$	$\frac{8}{5\pi}\left[\frac{\pi}{2}\lvert\mu\rvert - \lvert\mu\rvert\cos^{-1}(\lvert\mu\rvert) + \frac{1}{3}\sqrt{1-\mu^2}\,(2+\mu^2)\right]$
spherical-4	$5\cos^4\vartheta_{_L}$	$\frac{5}{48}(3+6\mu^2-\mu^4)$
spherical-5	$6\cos^5\vartheta_{_L}$	$\frac{12}{7\pi}\left[\frac{\pi}{2}\lvert\mu\rvert - \lvert\mu\rvert\cos^{-1}(\lvert\mu\rvert) + \frac{2}{15}\sqrt{1-\mu^2}\,(4+\frac{9}{2}\mu^2-\mu^4)\right]$
spherical-6	$7\cos^6\vartheta_{_L}$	$\frac{7}{128}(5+15\mu^2-5\mu^4+\mu^6)$

Applying the parameterisations (1) and assuming the separation (2) with $g_\varphi = 1$, (4) can rewritten as

$$G(\hat{\boldsymbol{\omega}}(\mu,\varphi)) = \frac{1}{2\pi}\int\limits_{[0,2\pi]}\int\limits_{[0,\frac{\pi}{2}]} \lvert\langle\,\hat{\boldsymbol{\omega}}(\mu,\varphi), \boldsymbol{\omega}(\vartheta_{_L},\varphi_{_L})\,\rangle\rvert\; g_\vartheta(\vartheta_{_L})\,\sin\vartheta_{_L}\,d\vartheta_{_L}\,d\varphi_{_L}. \quad (5)$$

It can be shown that $G_\mu(\mu) := G(\hat{\boldsymbol{\omega}}(\mu,\varphi))$ is independent of the azimuth angle φ of the radiation direction $\mathbf{y} = \hat{\boldsymbol{\omega}}(\mu,\varphi)$ for all standard LNDFs of Table 1 which also presents the respective explicit and analytical expressions of $G_\mu(\mu)$. For detailed information see Otto and Trautmann (2008a).

3 A generalised GF for a trigonometric LNDF

We now consider a trigonometric LNDF

$$\begin{aligned} g_\vartheta(\vartheta_{_L}) := a \;&+\; b\cos\vartheta_{_L} + c\cos 2\vartheta_{_L} + d\cos 3\vartheta_{_L} \\ &+\; e\cos 4\vartheta_{_L} + f\cos 5\vartheta_{_L} + g\cos 6\vartheta_{_L} \end{aligned} \quad (6)$$

with the free parameters $a, b, c, d, e, f, g \in \mathbb{R}$. All standard LNDFs considered by Otto and Trautmann (2008a) can be written in terms of this trigonometric representation. Beyond, measurements of LNDF can be fitted to this model, or above parameters can be chosen arbitrarily (that $g_\vartheta \geq 0$) to describe situations of realistic vegetation canopies. But note that the normalisation condition (3) applied to (6) leads to the constraint

$$1 = a + \frac{b}{2} - \frac{c}{3} - \frac{d}{2} - \frac{e}{15} + \frac{f}{6} - \frac{g}{35}. \tag{7}$$

Thus, one parameter can be expressed by the remaining free parameters.
In the following we wish to derive GF for our trigonometric LNDF. To do that we rewrite (6) in powers of $\cos\vartheta_L$ with the help of trigonometric theorems, that is,

$$\begin{aligned} g_\vartheta(\vartheta_L) = a \quad &+ \quad b\,\cos\vartheta_L + c\,(2\cos^2\vartheta_L - 1) + d\,(4\cos^3\vartheta_L - 3\cos\vartheta_L) \\ &+ \quad e\,(1 - 8\cos^2\vartheta_L + 8\cos^4\vartheta_L) \\ &+ \quad f\,(5\cos\vartheta_L - 20\cos^3\vartheta_L + 16\cos^5\vartheta_L) \\ &+ \quad g\,(18\cos^2\vartheta_L - 48\cos^4\vartheta_L + 32\cos^6\vartheta_L - 1). \end{aligned}$$

This leads to

$$\begin{aligned} g_\vartheta(\vartheta_L) = a - c + e - g \quad &+ \quad \cos\vartheta_L\,(b - 3d + 5f) \\ &+ \quad \cos^2\vartheta_L\,(2c - 8e + 18g) \\ &+ \quad \cos^3\vartheta_L\,(4d - 20f) \\ &+ \quad \cos^4\vartheta_L\,(8e - 48g) \\ &+ \quad \cos^5\vartheta_L\,16f \\ &+ \quad \cos^6\vartheta_L\,32g \end{aligned}$$

$$\begin{aligned} &= (a - c + e - g)\cdot 1 \\ &+\frac{1}{2}\,(b - 3d + 5f)\cdot 2\cos\vartheta_L \\ &+\frac{1}{3}\,(2c - 8e + 18g)\cdot 3\cos^2\vartheta_L \\ &+\frac{1}{4}\,(4d - 20f)\cdot 4\cos^3\vartheta_L \\ &+\frac{1}{5}\,(8e - 48g)\cdot 5\cos^4\vartheta_L \\ &+\frac{16}{6}\,f\cdot 6\cos^5\vartheta_L \\ &+\frac{32}{7}\,g\cdot 7\cos^6\vartheta_L \end{aligned} \tag{8}$$

in which the standard LNDFs as presented in Table 1 appear as the factors after each multiplication sign. Inserting (8) into (5) this allows to apply the standard GFs (Table 1) which leads to the GF for our trigonometric LNDF

$$
\begin{aligned}
G_\mu(\mu) &= (a-c+e-g)\cdot\frac{1}{2}\\
&+ \frac{1}{2}(b-3d+5f)\cdot\frac{4}{3\pi}\left[\frac{\pi}{2}|\mu| - |\mu|\,\cos^{-1}(|\mu|) + \sqrt{1-\mu^2}\right]\\
&+ \frac{1}{3}(2c-8e+18g)\cdot\frac{3}{8}(1+\mu^2)\\
&+ \frac{1}{4}(4d-20f)\cdot\frac{8}{5\pi}\left[\frac{\pi}{2}|\mu| - |\mu|\,\cos^{-1}(|\mu|) + \frac{1}{3}\sqrt{1-\mu^2}\,(2+\mu^2)\right]\\
&+ \frac{1}{5}(8e-48g)\cdot\frac{5}{48}(3+6\mu^2-\mu^4)\\
&+ \frac{16}{6}f\cdot\frac{12}{7\pi}\left[\frac{\pi}{2}|\mu| - |\mu|\,\cos^{-1}(|\mu|) + \frac{2}{15}\sqrt{1-\mu^2}\,(4+\frac{9}{2}\mu^2-\mu^4)\right]\\
&+ \frac{32}{7}g\cdot\frac{7}{128}(5+15\mu^2-5\mu^4+\mu^6)\\
&= \frac{1}{2}(a-c+e-g) + \frac{1}{8}(2c-8e+18g) + \frac{1}{16}(8e-48g) + \frac{5}{4}g\\
&+ \mu^2\left[\frac{1}{8}(2c-8e+18g) + \frac{1}{8}(8e-48g) + \frac{15}{4}g\right]\\
&+ \mu^4\left[-\frac{1}{48}(8e-48g) - \frac{5}{4}g\right]\\
&+ \mu^6\left[\frac{32}{128}g\right]\\
&+ |\mu|\left[\frac{1}{3}(b-3d+5f) + \frac{1}{5}(4d-20f) + \frac{16}{7}f\right]\\
&+ \frac{1}{\pi}|\mu|\cos^{-1}(|\mu|)\left[-\frac{2}{3}(b-3d+5f) - \frac{2}{5}(4d-20f) - \frac{32}{7}f\right]\\
&+ \frac{1}{\pi}\sqrt{1-\mu^2}\left[\frac{2}{3}(b-3d+5f) + \frac{4}{15}(4d-20f) + \frac{256}{105}f\right]\\
&+ \frac{1}{\pi}\mu^2\sqrt{1-\mu^2}\left[\frac{2}{15}(4d-20f) + \frac{288}{105}f\right]\\
&+ \frac{1}{\pi}\mu^4\sqrt{1-\mu^2}\left[-\frac{64}{105}f\right].
\end{aligned}
$$

Finally we obtain as in a recent paper (Otto and Trautmann, 2008b)

$$
\begin{aligned}
G_\mu(\mu) &= \frac{1}{4}(2a-c) + \frac{c}{4}\mu^2 - \frac{1}{2}\left(\frac{e}{3}+\frac{g}{2}\right)\mu^4 + \frac{g}{4}\mu^6 + \left(\frac{b}{3}-\frac{d}{5}-\frac{f}{21}\right)|\mu|\\
&- \frac{2}{\pi}\left(\frac{b}{3}-\frac{d}{5}-\frac{f}{21}\right)|\mu|\cos^{-1}(|\mu|) + \frac{2}{3\pi}\left(b-\frac{7}{5}d+\frac{23}{35}f\right)\sqrt{1-\mu^2}\\
&+ \frac{8}{15\pi}\left(d+\frac{f}{7}\right)\mu^2\sqrt{1-\mu^2} - \frac{64f}{105\pi}\mu^4\sqrt{1-\mu^2}.
\end{aligned}
$$

Applying this GF for the trigonometric LNDF this enables us to extend our two-stream methods for the standard LNDFs (Otto and Trautmann, 2008a) to more realistic canopy structures.

4 Outlook

We demonstrated that the GF can be calculated explicitly and analytically for a trigonometric LNDF which can be treated to be more representative for realistic vegetation canopies. With regard to analytical solutions of the radiative transfer equation for a turbid vegetation medium, this opens up the possibility to develop hierarchies of fast radiative transfer solvers, for example based on the two-stream approximation, considering realistic leaf architectures. The adopted trigonometric LNDF can be seen as a promising generalisation of the commonly used standard LNDFs and allows to extend our analytical two-stream methods (Otto and Trautmann, 2008a) beyond these standard cases. We also hope to apply this knowledge to four- and multi-stream considerations.

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